# Total variation distance estimates for stochastic differential delay equations with small noises

Nguyễn Thu Hằng

Đại học Mỏ - Địa chất

Hội nghị Khoa học Khoa Toán - Cơ - Tin học
Hà Nội, 10/2024

# Content

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1. Introduction
2. Main results

# 1. Introduction

We consider the stochastic differential delay equation with small noise of the form

$$\begin{cases} X_{\varepsilon,t} = \varphi(0) + \int_0^t b(X_{\varepsilon,s}, X_{\varepsilon,s-\tau})ds + \varepsilon \int_0^t \sigma(X_{\varepsilon,s}, X_{\varepsilon,s-\tau})dB_s, \ \ t \in [0,T] \\ X_{\varepsilon,t} = \varphi(t), \ \ t \in [-\tau, 0], \end{cases} \tag{1}$$

where the initial data $\varphi : [-\tau, 0] \to \mathbb{R}$ is a bounded deterministic function, $(B_t)_{t\in[0,T]}$ is a standard Brownian motion and $b, \sigma$ are deterministic functions on $\mathbb{R}^2$.

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Intuitively, as $\varepsilon$ tends to 0, $(X_{\varepsilon,t})_{t\geq 0}$ the solution to (1) tends to $(x_t)_{t\geq 0}$, which solves the following deterministic differential delay equations

$$\begin{cases} x_t = \varphi(0) + \int_0^t b(x_s, x_{s-\tau})ds, \; t \in [0,T] \\ x_t = \varphi(t), \; t \in [-\tau, 0]. \end{cases} \tag{2}$$

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$$\begin{cases} Y_t = \int_0^t \left(b_1'(x_s, x_{s-\tau})Y_s + b_2'(x_s, x_{s-\tau})Y_{s-\tau}\right) ds + \int_0^t \sigma(x_s, x_{s-\tau})dB_s, \ t \in [0, T] \\ Y_t = 0, \ t \in [-\tau, 0]. \end{cases} \tag{4}$$

We observe that, for each $t \in [0, T]$, $Y_t$ is a normal random variable.

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- Similar to [12] we have $\tilde{X}_{\varepsilon,t}$ converges to $Y_t$ in $L^p(\Omega), p \geq 2$ as $\varepsilon \to 0$.
- Thus, the sequence $(\tilde{X}_{\varepsilon,t})_{\varepsilon\in(0,1)}$ satisfies the central limit theorem as $\varepsilon \to 0$.

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▶ An important problem arising here is to investigate the rate of convergence via certain distances.

# 1. Introduction

There are three distances commonly used in the literature.
(i) The Wasserstein distance between the laws of $\tilde{X}_{\varepsilon,t}$ and $Y_t$ :

$$d_{\mathrm{W}}(\tilde{X}_{\varepsilon,t}, Y_t) := \sup_{|g(x)-g(y)|\leq|x-y|} |Eg(\tilde{X}_{\varepsilon,t}) - Eg(Y_t)|.$$

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(iii) The total variation distance between the laws of $\tilde{X}_{\varepsilon,t}$ and $Y_t$ :

$$\begin{aligned} d_{\mathrm{TV}}(\tilde{X}_{\varepsilon,t}, Y_t) &:= \sup_{A\in\mathcal{B}(\mathbb{R})} |P(\tilde{X}_{\varepsilon,t} \in A) - P(Y_t \in A)| \\ &= \frac{1}{2} \sup_{\|g\|_\infty\leq 1} |Eg(\tilde{X}_{\varepsilon,t}) - Eg(Y_t)|, \end{aligned}$$

where $\mathcal{B}(\mathbb{R})$ is Borel $\sigma$-algebra on $\mathbb{R}$ and $\|g\|_\infty = \sup_{x\in\mathbb{R}} |g(x)|$.

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Thus, we focus on bounding the total variation distance $d_{\mathrm{TV}}(\tilde{X}_{\varepsilon,t}, Y_t)$.

Our main tools are the techniques of Malliavin calculus and the following general result ( Theorem 3.1 in the recent paper [5].)

## Lemma 3.1.

*Let $F_1 \in \mathbb{D}^{2,4}$ be such that $\|DF_1\|_{L^2[0,T]} > 0$ a.s. Then, for any random variable $F_2 \in \mathbb{D}^{1,2}$ and any measurable function $g$ with $\|g\|_\infty = \sup_{x\in\mathbb{R}} |g(x)| \leq 1$, we have*

$$|Eg(F_1) - Eg(F_2)| \leq C\left(E\|DF_1\|^{-8}_{L^2[0,T]} E\left(\int_0^T\int_0^T |D_\theta D_r F_1|^2 d\theta dr\right)^2 + (E\|DF_1\|^{-2}_{L^2[0,T]})^2\right)^{\frac{1}{4}} \|F_1 - F_2\|_{1,2}, \tag{5}$$

*provided that the expectations exist, where $C$ is an absolute constant.*

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# 2. The main results

**Assumption 2.1.**

*$b, \sigma : \mathbb{R}^2 \to \mathbb{R}$ are twice differentiable functions with the partial derivatives bounded by $L$.*

Our Assumption 2.1 is slightly stronger than the conditions required in [12].

# 2. The main results

**Theorem 2.1.**

*Let Assumption 2.1 hold. Consider the stochastic processes* $(\tilde{X}_{\varepsilon,t})_{-\tau\le t\le T}$ *and* $(Y_t)_{-\tau\le t\le T}$ *defined by (3) and (4), respectively. Then, we have*

$$d_{\mathrm{TV}}(\tilde{X}_{\varepsilon,t}, Y_t) \le \frac{Ct\varepsilon}{\sqrt{\mathrm{Var}(Y_t)}} \ \forall \varepsilon \in (0,1), 0 < t \le T, \tag{6}$$

*where* $C$ *is a positive constant not depending on* $t$ *and* $\varepsilon$*.*

We also show that the convergence rate is of optimal order.

# 2. The main results

**Theorem 2.2.**

*Let Assumption 2.1 hold. We additionally assume that the partial derivatives of b and $\sigma$ are continuous. Then, for any continuous and bounded function $g$, we have*

$$\lim_{\varepsilon\to 0}\frac{Eg(\tilde{X}_{\varepsilon,t})-Eg(Y_t)}{\varepsilon}=\frac{1}{2\mathrm{Var}(Y_t)}E\left[g(Y_t)\delta\left(Z_tDY_t\right)\right],\ 0<t\le T, \tag{7}$$

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*where $Z_t=0$ for $t\in[-\tau,0]$ and*

$$\begin{aligned}Z_t&=\int_0^t b_1'(x_s,x_{s-\tau})Z_s ds+\int_0^t b_2'(x_s,x_{s-\tau})Z_{s-\tau}ds\\&+\int_0^t\left(b_{11}''(x_s,x_{s-\tau})Y_s^2+b_{12}''(x_s,x_{s-\tau})Y_sY_{s-\tau}+b_{21}''(x_s,x_{s-\tau})Y_sY_{s-\tau}+b_{22}''(x_s,x_{s-\tau})Y_{s-\tau}^2\right)ds\\&+2\int_0^t\left(\sigma_1'(x_s,x_{s-\tau})Y_s+\sigma_2'(x_s,x_{s-\tau})Y_{s-\tau}\right)dB_s,\ t\in[0,T].\end{aligned} \tag{8}$$

# 3. The main results

In particular, we have

$$\lim_{\varepsilon \to 0} \frac{d_{\mathrm{TV}}(\tilde{X}_{\varepsilon,t}, Y_t)}{\varepsilon} \geq \frac{1}{2\mathrm{Var}(Y_t)} E\left|E\left[\delta(Z_t DY_t)|Y_t\right]\right|, \ 0 < t \leq T. \qquad (9)$$

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Thank you for your listening!